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# MULTIPLICATIVE COMMUTATORS IN DIVISION RINGS

BY

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ABSTRACT

In this paper we study division rings in which the multiplicative commutators are periodic or periodic relative to the center.

As a consequence of one of their principal theorems in [4], Herstein, Procesi and Schacher proved that if $D$ is a division ring with center $Z$ such that $(xy - yx)^{n(x,y)} \in Z$, $n(x, y) \geqq 1$, for all $x, y \in D$ then $\dim_Z D \leqq 4$ (and so, in particular, $(xy - yx)^2 \in Z$ for all $x, y \in D$).

It seems rather natural to consider the multiplicative analog of the above-cited result, namely, to consider a division ring $D$ in which $(xyx^{-1}y^{-1})^{n(x,y)} \in Z$, $n(x, y) \geqq 1$, for all $x \neq 0$ and $y \neq 0$ in $D$. A very partial, fragmentary result in this direction was obtained by Putcha and Yaqub in [6]; they considered the situation where $n(x, y) = 2^{m(x,y)}$. Related to these kinds of things is a result by Zalesskii [9] on torsion normal subgroups of division rings. He showed that a periodic normal subgroup of a division ring must be central. We shall extend this, as a consequence of what we do here, to periodic subnormal subgroups of division rings.

The natural conjecture would be that a division ring $D$ in which $(xyx^{-1}y^{-1})^{n(x,y)} \in Z$ for all non-zero $x$ and $y$ in $D$ must be a field. We don't quite obtain this full result; but the results we do get imply those already known. We also show the result to be correct if $(xyx^{-1}y^{-1})^{n(x,y)} = 1$.

We shall use the following notation throughout. $Z$ will denote the center of

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the division ring $D$, and if $D_1$ is a subdivision ring of $D$ then $Z(D_1)$ will denote the center of $D_1$. If $x \in D$ then $C_D(x) = \{y \in D \mid xy = yx\}$, If $a \in D$ then $Z(a)$ will be the subfield of $D$ obtained by adjoining $a$ to $Z$.

We begin the material of this paper with

THEOREM 1. *Let $D$ be a division ring in which, for all $x \neq 0$ and $y \neq 0$ in $D$, $(xyx^{-1}y^{-1})^{n(x,y)} = 1$ for some $n(x,y) \geqq 1$. Then $D$ is commutative.*

PROOF. If all $xyx^{-1}y^{-1}$ are in $Z$ then, for $y \in D$, $xZ(y)x^{-1} \subset Z(y)$ for all $x \neq 0 \in D$. By the Brauer–Cartan–Hua theorem we have, since $Z(y)$ is commutative, that $Z(y) = Z$ and so $y \in Z$. Thus $D$ would be commutative.

Suppose then that $a = xyx^{-1}y^{-1} \neq 1$ for some $x, y \in D$. By hypothesis, $a^n = 1$ for some $n > 1$, hence $Z(a)$ is a normal extension of $Z$. Let $\phi \neq 1$ be an automorphism of $Z(a)$ over $Z$. By the Skolem–Noether theorem, $\phi(a) = bab^{-1} = a^i \neq a$ for some $b \in D$. Because $Z(a)$ is a finite extension of $Z$, $\phi^k = 1$ for some $k > 1$ hence $b^k a = ab^k$.

Let $D_1 = C_D(b^k)$; since $a, b \in D_1$ and $a^n = 1$, $b^k \in Z_1 = Z(D_1)$ and $ba = a^i b$, $a$ and $b$ generate a subdivision algebra $D_2$ of $D_1$ which is finite-dimensional over $Z_1$. Let $b^k = \lambda \in Z_1$; thus $c = b(a+\lambda)b^{-1}(a+\lambda)^{-1} = (a^i + \lambda)/(a+\lambda) \neq 1$ is a commutator in $D_2$, whence $c^m = 1$ for some $m > 1$. Since $ac = ca$ and $\lambda = (a^i + ac)/(c-1)$ and $a$, $c$ are roots of unity, we obtain that $\lambda$ is obtained from the prime field $P$ by adjunction of roots of unity. Hence $\lambda$ is algebraic over $P$.

If $\operatorname{Char} D = p \neq 0$, this last statement implies that $P(\lambda)$ is a finite field, hence $1 = \lambda^t = b^{kt}$ for some $t \geqq 1$, that is, $b$ is of finite period. Since $a$ is also of finite period and $ba = a^i b$ we have that $a$ and $b$ generate a finite division ring $E$ over $P$. By Wedderburn's theorem, $E$ is a field. However, $a$ and $b$ are both in $E$, in consequence of which $ab = ba$. This contradicts that $ba = a^i b \neq ab$.

So we may assume that $\operatorname{char} D = 0$; hence $\lambda$ is algebraic over the rational field $P$, and so $b$ is algebraic over $P$. From this fact and the relations $a^m = 1$ and $ba = a^i b$ we obtain that $a$ and $b$ generate a division algebra $D_3$ over $P$ which is finite-dimensional over $P$. But then, from the properties of the roots of unity in a finite extension of the rationals, we have that roots of unity in $D_3$ are of bounded order. Therefore there exists an integer $N > 0$ such that $(uvu^{-1}v^{-1})^N = 1$ for all $u \neq 0$, $v \neq 0$ in $D_3$. By a result of Amitsur [1, theor. 19], $D_3$ is commutative. Since $a, b \in D_3$ we have that $ab = ba$ in contradiction to $ab \neq ba$. In this way we have shown that $D$ is a field, and so Theorem 1 is proved.

If $D$ is a division ring finite-dimensional over its center $Z$ and if $y \in D$ is a product of commutators $u_i v_i u_i^{-1} v_i^{-1}$, then if $y^m = \lambda \in Z$ we must have that $\lambda$ is a root of unity. For if $N$ is the norm on $D$ to $Z$, since $N(uvu^{-1}v^{-1}) = 1$, we have

$N(y)=1$ and $N(\lambda)=\lambda^t$ where $t=\dim_Z D$; thus since $y^n=\lambda$, $1=N(y^n)=N(\lambda)=\lambda^t$. Since we shall use this remark several times we record this as a

SUBLEMMA. *If $D$ is a division ring finite-dimensional over $Z$ and if $y\in D$ is a product of multiplicative commutators then, if $y^n=\lambda\in Z$, we must have that $\lambda$ is a root of unity.*

The Sublemma together with Theorem 1 immediately imply

THEOREM 2. *Let $D$ be a division ring finite-dimensional over its center $Z$, such that $(xyx^{-1}y^{-1})^{n(x,y)}\in Z$, $n(x,y)\geqq 1$, for all non-zero $x$ and $y$ in $D$. Then $D$ is commutative.*

We shall now extend Theorem 2, dropping the assumption of the finite-dimensionality of $D$ over $Z$. We are not able to get the full general theorem we seek — for we still need to impose a condition on $D$ — but the result we obtain is strong enough to derive readily the known results in this direction.

THEOREM 3. *Let $D$ be a division ring in which for all non-zero $x, y$ in $D$, $(xyx^{-1}y^{-1})^{n(x,y)}\in Z$ for some $n(x,y)\geqq 1$. If $a\in D$, $a\notin Z$ is algebraic over $Z$ then $Z(a)$ admits no non-trivial automorphism over $Z$.*

PROOF. Suppose that $\phi\neq 1$ is an automorphism of $Z(a)$ over $Z$. Since $a$ is algebraic over $Z$ we must have that $\phi^k=1$ for some $k>1$.

By the Skolem–Noether theorem there exists an $x\in D$ such that $\phi(a)=xax^{-1}$; therefore $x^ka=ax^k$. Now, both $a$ and $x$ are in $C_D(x^k)$ and both are algebraic over $Z_1=Z(C_D(x^k))$. Because $xax^{-1}=q(a)\neq a$ where $q(a)\in Z(a)$, we have that $x$ and $a$ generate a finite-dimensional division algebra $D_1$ over $Z_1$. By Theorem 2 we must have that $D_1$ is commutative. However, since both $x$ and $a$ are in $D_1$ and do not commute, we have a contradiction. The theorem is now proved.

As a very special case of Theorem 3 we have

THEOREM 4. *If $D$ is a division ring in which $(xyx^{-1}y^{-1})^{n(x,y)}\in Z$, $n(x,y)\geqq 1$, for all $x\neq 0$, $y\neq 0$ in $D$, if $a^2\in Z$ then $a\in Z$.*

PROOF. Suppose that $a\notin Z$; then for some $x\in D$, $u=xax^{-1}a^{-1}\neq 1$. Since $a^2\in Z$, $(ua)^2=xa^2x^{-1}=a^2$ from which we get $uau=a$, and so $aua^{-1}=u^{-1}$. Since $u$ is a commutator, $u^n\in Z$ for some $n>0$, so $u$ is certainly algebraic over $Z$. If $u^{-1}=u$ then $u^2=1$, and since $u\neq 1$ we have that $u=-1$. This would give us that $xax^{-1}=a\neq a$ and so $Z(a)$ admits the non-trivial automorphism, induced by conjugation by $x$, over $Z$. By Theorem 3 this is not possible. Hence $u^{-1}\neq u$.

Since $u^{-1} = aua^{-1}$, $Z(u)$ admits the non-trivial automorphism over $Z$ induced by conjugation by $a$. By Theorem 3, this is impossible.

An immediate corollary to this last theorem is the

COROLLARY 1. *If $D$ is a division ring in which $(xyx^{-1}y^{-1})^{n(x,y)} \in Z$ for all $x \neq 0$, $y \neq 0$ in $D$ then $(xyx^{-1}y^{-1})^{m(x,y)} \in Z$ where $m(x,y)$ is odd.*

From this corollary we read off trivially the special result by Putcha and Yaqub cited earlier [6].

COROLLARY 2. *If $D$ is a division ring in which $(xyx^{-1}y^{-1})^{2^{n(x,y)}} \in Z$ for all $x \neq 0$, $y \neq 0$ in $D$, then $D$ is commutative.*

We can drop the algebraic condition on $a$ in Theorem 3, by restating the result in a somewhat different form.

THEOREM 5. *Let $D$ be a division ring in which $(xyx^{-1}y^{-1})^{n(x,y)} \in Z$ for all non-zero $x$ and $y$ in $D$, with $n(x,y) \geqq 1$. If $a, b \in D$ are such that $bab^{-1} \in C_D(a)$ then $ab = ba$.*

PROOF. Suppose that $ab \neq ba$. If $u = bab^{-1}a^{-1}$ then, if $u = \lambda \neq 1 \in Z$, we would have that $bab^{-1} \in Z(a)$, and so $bZ(a)b^{-1} \subset Z(a)$. If $v = b(a+1)b^{-1}(a+1)^{-1} = (\lambda a + 1)/(a+1)$, since $v$ is a commutator, $v^m \in Z$ for some $m \geqq 1$. Hence $[(\lambda a + 1)/(a+1)]^m = \beta \in Z$. If $\beta \neq \lambda^m$, this last relation would force $a$ to be algebraic over $Z$. This, together with $bZ(a)b^{-1} \subset Z(a)$ would yield, by applying Theorem 3, that $ba = ab$. Hence we must assume that $\beta = \lambda^m$, and so $[(\lambda a + 1)/(a+1)]^m = \lambda^m$. In other words $[(\lambda a + 1)/\lambda(a+1)]^m = 1$. Let $c = (\lambda a + 1)/\lambda(a+1)$; since $\lambda \neq 1$ and $a \in Z$ we immediately know that $c \in Z$. But $c^m = 1$, $c \notin Z$, hence $Z(c)$ admits a non-trivial automorphism over $Z$. By Theorem 3 this is not possible. So we conclude that $u = bab^{-1}a^{-1}$ is *not* in $Z$.

Since $u$ is a commutator, $u^n = \gamma \in Z$ for some $n > 1$. Because $bab^{-1} \in C_D(a)$ by hypothesis, $\gamma = u^n = (bab^{-1}a^{-1})^n = (bab^{-1})^n a^{-n} = ba^n b^{-1} a^{-n}$. By the reasoning of the paragraph above we must have $\gamma = 1$ and so $u^n = \gamma = 1$. Since $u \notin Z$ is a root of unity, $Z(u)$ admits a non-trivial automorphism over $Z$. By Theorem 3 this is not possible. In consequence, $ab = ba$ and so the theorem is proved.

We now pass to a local version of Theorem 4. This is

THEOREM 6. *Let $D$ be a division ring, and let $a \in D$ be such that $(axa^{-1}x^{-1})^{n(x)} \in Z$, $n(x) \geqq 1$, for all $x \neq 0$ in $D$. If $a^2 \in Z$ then $a \in Z$.*

PROOF. Suppose that $a \notin Z$. From the Brauer–Cartan–Hua theorem we immediately have that $u = axa^{-1}x^{-1} \notin Z$ for some $x \in D$. As in the proof of Theorem 4 we obtain that $ua^{-1}u = a^{-1}$, and so $aua^{-1} = u^{-1} \neq u$.

Consider $Z(u)$; $a$ induces an automorphism of period 2 on $Z(u)$ by conjugation. Hence the fixed field $F$ of this automorphism is of co-dimension 2 in $Z(u)$. Therefore $Z(u) = \{\alpha + \beta u \mid \alpha, \beta \in F\}$.

If $\alpha, \beta \in F$ they commute with $a$, whence $y = a(\alpha + \beta u)a^{-1}(\alpha + \beta u)^{-1} = (\alpha + \beta u^{-1})/(\alpha + \beta u) = \gamma(\alpha + \beta u)^{-2}$, where $\gamma = (\alpha + \beta u^{-1})(\alpha + \beta u)$ is in $Z$. Since $y$ is a commutator of $a$, $y^n \in Z \in F$ for some $n \geqq 1$, hence $(\alpha + \beta u)^{2n} \in F$ for all $\alpha, \beta \in F$. If $t \in Z(u)$, $t = \alpha + \beta u$ for some $\alpha, \beta \in F$, thus $t^{m(t)} \in F \neq Z(u)$ for some $m(t) \geqq 1$. By a result of Kaplansky [5], $F$ is of characteristic $p$ different from 0, and either $Z(u)$ is purely inseparable over $F$ of $Z(u)$ is algebraic over the prime field having $p$ elements.

Now $u^2 - \lambda u + 1 = 0$ where $\lambda = u + u^{-1} \in F$. So, if $\operatorname{char} F \neq 2$, $u$ is separable over $F$. If $\operatorname{char} F = 2$, then $u$ is inseparable over $F$ only if $\lambda = 0$, and so, only if $u^2 = 1$. But this would force $u = 1$, contrary to $u \neq 1$. Hence $Z(u)$, and so $Z$, must be algebraic over the prime field. Since $a^2 \in Z$, $a^2$ is algebraic over $GF(p)$ and so $a^m = 1$ for some $m \geqq 1$. Because $u \in Z(u)$, $u$ is algebraic over $GF(p)$, hence $u^n = 1$ for some $n \geqq 1$. These two relations on $u$ and $a$ together with $aua^{-1} = u^{-1}$ give us that $a$ and $u$ generate a finite division ring. By Wedderburn's theorem we conclude that $au = ua$, contrary to $au \neq ua$. The theorem is thereby proved.

At this point it seems appropriate to make two conjectures which touch on the things we have been discussing. The first is a local version of Theorem 1; the second is a local version of the general result we seek here. Of course the truth of the second conjecture would imply that of the first one.

CONJECTURE 1. *Let $D$ be a division ring, $a \in D$ such that $(axa^{-1}x^{-1})^{n(x)} = 1$ for all $x \neq 0 \in D$. Then $a \in Z$.*

CONJECTURE 2. *Let $D$ be a division ring, $a \in D$ such that $(axa^{-1}x^{-1})^{n(x)} \in Z$ for all $x \neq 0 \in D$. Then $a \in Z$.*

We continue studying division rings in which $(xyx^{-1}y^{-1})^{n(x,y)} \in Z$. We prove

LEMMA 1. *Let $D$ be a division ring in which $(xyx^{-1}y^{-1})^{n(x,y)} \in Z$ for all $x \neq 0$, $y \neq 0$ in $D$. Then $D$ contains no elements purely inseparable over $Z$.*

PROOF. Suppose that $\operatorname{char} D = p \neq 0$ and $a \notin Z$ but $a^p \in Z$. We follow a familiar route. Let $[u, v] = uv - vu$ and $[u_1, u_2, \cdots, u_n] = [[u_1, \cdots, u_{n-1}], u_n]$. Thus for any $x \in D$,

$$[x, \underbrace{a, a, \cdots, a}_{p\text{-times}}] = 0.$$

So there exists a $y \in D$ with $ay - ya \neq 0 \in C_D(a)$. As usual this gives an element $b \in D$ such that $ab - ba = a$, and so $aba^{-1} = 1 + b$. Thus $aba^{-1}b^{-1} = 1 + b^{-1}$, and since $(aba^{-1}b^{-1})^k \in Z$ we have $(1 + b^{-1})^k \in Z$. Therefore $b$ is algebraic over $Z$. So, since $aba^{-1} = 1 + b$, $Z(b)$ admits a non-trivial automorphism over $Z$. By Theorem 3, this situation is not possible, whereby the lemma is proved.

With the lemma in hand we can prove

THEOREM 7. *Let $D$ be a division ring in which $(xyx^{-1}y^{-1})^{n(x,y)} \in Z$ for all $x \neq 0$, $y \neq 0$ in $D$. Then any element in $D$ which is algebraic over $Z$ is separable over $Z$.*

PROOF. Suppose that $a \in D$, $a \notin Z$ is inseparable over $Z$. Therefore $a^{p^n}$ is separable over $Z$ for some $n$, where $p = \operatorname{char} D \neq 0$. By Lemma 1, $a^{p^n} \notin Z$. Consider $C_D(a^{p^n})$. In it, $a$ is purely inseparable over $Z(C_D(a^{p^n}))$, so, by Lemma 1, we have that $C_D(a^{p^n})$ is commutative, unless $a \in Z(C_D(a^{p^n}))$. But if $a \in Z(C_D(a^{p^n})) = Z(a^{p^n})$ (by the double centralizer theorem) then $a$ is separable over $Z$ since $Z(a^{p^n})$ is a separable extension of $Z$. On the other hand, if $C_D(a^{p^n})$ is commutative, it is a maximal subfield of $D$ and, by the double centralizer theorem, $C_D(a^{p^n}) = Z(a^{p^n})$. Since $a \in C_D(a^{p^n}) = Z(a^{p^n})$, as above we conclude that $a$ is separable over $Z$. At any rate, in all cases $a$ turns out to be separable over $Z$, which is precisely the contention of the theorem.

We turn to a question which is a bit off the direction we have followed until now, but is closely related in spirit to the results we have established. The next theorem extends a result of Zalesskii [9], who proved it for periodic normal subgroups of $D$.

THEOREM 8. *Let $D$ be a division ring and let $N$ be a subnormal subgroup of the multiplicative group of $D$. If $N$ is periodic then $N \subset Z$.*

PROOF. It is enough to show that $N$ is commutative, for if $K$ is the subdivision ring of $D$ generated by $N$ over $Z$ then $aKa^{-1} \subset K$ for all $a \in N$. If $N \not\subset Z$, then by a result of Stuth [8] either $K \subset Z$ or $K = D$. If $N$ is commutative, then $K$ is a field so we obtain, from the above, that $K \subset Z$. Since $N \subset K$ we derive that $N \subset Z$. So, to prove the theorem, we must merely show that $N$ is commutative.

We first dispose of the case $\dim_Z D < \infty$. In that case $N$ is isomorphic to a torsion group of matrices, so by the affirmative answer to the Burnside Problem in this case $N$ must be locally finite. If $\operatorname{Char} D = p \neq 0$ it then follows from a result of Herstein [2] that $N$ is abelian, so by the argument of the first paragraph, $N \subset Z$. Hence we may assume that $\operatorname{Char} D = 0$.

Suppose that $N$ is not commutative, and that $a, b \in N$ are such that $ab \neq ba$. Consider $N_0$, the group generated by $a$ and $b$. By the local finiteness of $N$, $N_0$ is a finite group. So $D_1 = QN_0$, the span of $N_0$ over the rational field $Q$, is a division algebra finite-dimensional over $Q$. Also, $M = D_1 \cap N \supset N_0$ is a subnormal subgroup of $D_1$, and is clearly a torsion group. If $M$ is abelian, because $a, b \in N_0 \subset M$ we have the contradiction $ab = ba$.

So we may assume that $D$ is finite-dimensional over $Q$, and that $N$ is a non-commutative, periodic subnormal subgroup of $D$. As a torsion group of matrices over $Q$, by a theorem of Schur [7], $N$ has an abelian normal subgroup $A$ of finite index in $N$. The subdivision ring generated by $A$ is a field $K$ such that $nKn^{-1} \subset K$ for all $n \in N$. Since $N \not\subset Z$, by the previously-quoted result of Stuth, we have that $A \subset Z$.

Now $Z$ is a finite extension of $Q$, so has only a finite number of roots of unity. Yet every $a \in A$ is such a root of unity. The net result of this is that $A$ is a finite group. Since $A$ is of finite index in $N$, we conclude that $N$ is a finite group.

If $N$ is normal in $D$, then every element of $N$ has all its conjugates in $N$, therefore has only a finite number of conjugates. By a theorem of Herstein [3] we get that $N \subset Z$. So we may assume that $N$ is not normal in $D$.

Since $N$ is subnormal in $D$, $N \lhd N_1 \lhd N_2 \lhd \cdots N_k \lhd D$. If $t \in N_1$ then $t$ induces an automorphism on the finite group $N$ by conjugation. Thus we have that $t^r \in C_D(N)$, where $r = (o(N))!$. However, since $N \not\subset Z$ and $xC_D(N)x^{-1} \subset C_D(N)$ for all $x \in N$, by Stuth's theorem we have that $C_D(N) = Z$ or $C_D(N) = D$. The latter possibility implies that $N \subset Z$. Therefore $C_D(N) \subset Z$. Since $t^r \in C_D(N)$, we have that $t^r \in Z$.

Let $B$ be the commutator subgroup of $N_1$. If $b \in B$, by the above, $b^r = \beta \in Z$. By the Sublemma we therefore have that $\beta$ is a root of unity. Thus $b^m = 1$ for $b \in B$. Since $B \lhd N_2 \lhd \cdots N_k \lhd D$, using an induction on the length of subnormality (we already have the result, above, for normal subgroups) we get that $B \subset Z$. So, if $a, b \in N \subset N_1$, then $aba^{-1}b^{-1} = \alpha \in Z$ and so $aba^{-1} \in Z(b)$ for every $a \in N$. Thus $aZ(b)a^{-1} \subset Z(b)$ for every $a \in N$; since $N \not\subset Z$, by the theorem of Stuth we conclude that $Z(b) = Z$ and so $b \in Z$. Thus $N \subset Z$, and the result is proved if $\dim_Z D < \infty$.

Suppose, then, that $\dim_Z D = \infty$, and that $N$ is a subnormal periodic subgroup of $D$. Suppose that $N \not\subset Z$, and let $a \in N$, $a \notin Z$. Since $a^k = 1$ for some $k > 1$, the field $Z(a)$ admits an automorphism $\phi \neq 1$ over $Z$, and $\phi(a) = a^i \neq a$. By the Skolem–Noether theorem there is an $x \in D$ such that $xax^{-1} = \phi(a) = a^i$. Since $Z(a)$ is finite-dimensional over $Z$, $\phi^t = 1$ and so $x^t a = ax^t$ for some $t \geqq 1$. Let $D_0$ be the subdivision ring generated by $x$ and $a$ over $Z$. From the fact that $x^t$

commutes with both $x$ and $a$ we have that $x^t \in Z_0 = Z(D_0)$. Because $a^k = 1$ and $xax^{-1} = a^i$, we then have that $D_0$ is finite-dimensional over $Z_0$. Since $N_0 = N \cap D_0$ is a periodic subnormal sugroup of $D_0$, which is finite-dimensional over its center $Z_0$, the result we have established above in the finite-dimensional case tells us that $N_0 \subset Z_0$. However, $a \in N_0$ and $a \not\in Z_0$ since $x \in D_0$ and $ax \neq xa$. With this contradiction the theorem is proved.

It seems reasonable that the generalization of Theorem 8 from torsion groups to groups which are torsion with respect to $Z$ might be true. This is

CONJECTURE 3. *Let $D$ be a division ring and let $N$ be a subnormal subgroup of $D$. If $a^{n(a)} \in Z$ for every $a \in N$, where $n(a) > 0$, then $N \subset Z$.*

We finish this paper with a result which gives some information in the direction of verifying Conjecture 3.

THEOREM 9. *Let $D$ be a division ring and $N$ a subnormal subgroup of $D$ such that $a^{n(a)} \in Z$ for all $a \in N$, where $n(a) > 0$. Then every element in $N$ of finite period must be in $Z$.*

PROOF. Suppose that $a \in N$, $a \not\in Z$ is such that $a^m = 1$ for $m > 0$. By the Skolem–Noether theorem there exists an $x \in D$ such that $xax^{-1} = a^i \neq a$. Thus $x^t a = ax^t$ for some $t > 0$. Therefore the subdivision ring $D_1$ generated by $a$ and $x$ over $Z$ is finite-dimensional over $Z_1 = Z(D_1)$. If $N_1 = N \cap D_1$, then $N_1$ is subnormal in $D_1$ and $b^{n(b)} \in Z_1$ for all $b \in N_1$.

Let $B$ be the commutator subgroup of $N_1$. If $b \in B$ then $b^{n(b)} = \beta \in Z_1$; by the Sublemma, $\beta$ is a root of unity. Hence $b^{m(n)} = 1$ for $m(b) > 0$. So $B$ is a periodic subnormal subgroup of $D_1$. By Theorem 8, $B \subset Z_1$. So, if $u, v \in N_1$ then $uvu^{-1}v^{-1} \in B \subset Z_1$, hence $uvu^{-1} = \lambda v$ where $\lambda \in Z_1$. Thus $u \in Z(v)u^{-1} \subset Z(v)$ for all $u \in N_1$. Since $a \in N_1$ and $a \not\in Z_1$, $N_1 \not\subset Z_1$. Therefore, by Stuth's theorem, $Z(v) = Z$ for all $v \in N_1$, and so $N_1 \subset Z_1$. This contradicts $N_1 \not\subset Z_1$, and proves the theorem.

If $D$ is a division ring of characteristic $p \neq 0$ and $N$ is a subnormal subgroup of $D$ such that $x^{n(x)} \in Z$ for every $x \in N$, using the argument of Lemma 1 in conjunction with Theorem 8 it is easy to show that every element $a \in N$ such that $a^{p^{n(a)}} \in Z$ must be in $Z$. In particular if $N$ is such that $x^{p^{n(x)}} \in Z$ for all $x \in N$ then $N$ is in the center.

*Added in Proof.* It has come to my attention that Monastyrni (Math. Rev. **51**, 5782) has proved a slight generalization of my Theorem 8, and deserves priority on this result of Theorem 8.

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